



Robust saturation controller for linear time-invariant system with structured real parameter uncertainties

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Abstract

This paper is focused on a robust saturation controller for the linear time-invariant (LTI) system involving both actuator's saturation and structured real parameter uncertainties. The controller suggested in this paper can analytically prescribe the upper and lower bounds of parameter uncertainties, and guarantee the closed-loop robust stability of the system in the presence of actuator's saturation. The suboptimal bang–bang control method is extended to LTI system with parameter uncertainties. Based on affine quadratic stability and multi-convexity concept, the *robust optimal bang–bang controller* is newly derived by minimizing the time derivative of affine Lyapunov function subjected to the limit of control force. Since this controller is a gain-scheduled type, it requires the exact knowledge of uncertain parameters. Another robust saturation controller with a fixed gain is proposed and the *linear matrix inequality (LMI)-based* sufficient existence conditions for a fixed-gain controller are derived. The effectiveness and the availability of the proposed controller are investigated by a practical numerical example. Through numerical simulations, it is confirmed that the proposed robust saturation controller is robustly stable with respect to parameter uncertainties over the prescribed range defined by the upper and lower bounds.

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1. Introduction

Most of the actuation devices are subject to magnitude saturation. The physical inputs such as force, torque, thrust, stroke, voltage, current, and flow rate of all conceivable applications of current technology are ultimately limited. Unexpected large amplitude disturbances can push systems' actuators into saturation, thus forcing the system to operate in a nonlinear mode for which it was not designed and in which it may be unstable [1]. In recent years, research on the active control of civil engineering structures such as bridges and buildings has received increasing attention [2–4]. Many control strategies have been developed with the goal of protecting buildings subjected to weak and moderate earthquakes. But one of the main difficulties in realizing the active control systems to protect the building against strong earthquakes is the demand of unrealistic large

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control force. With the linear active controller such as linear quadratic regulator (LQR) which has been used extensively in many structural control applications, the applied control force is a linear combination of the structural responses. Therefore, the maximum control force will correspond to the peak response which actually occurs only for a few times during strong earthquakes. Thus the design of an actuator based on the maximum demand of control force is inefficient and uneconomical. Saturation control considering the limit of control force is known to be able to embrace this problems and, furthermore, to be more effective in the reduction of structural response than the linear control algorithms under strong earthquakes [5–9]. Most of the existing saturation control algorithms are developed in nominal linear time-invariant (LTI) system. Because inherent modeling errors between mathematical models and real-world systems are unavoidable, active controller for systems with actuator's saturation should be designed to be robust with respect to system uncertainties.

The bang–bang control, which minimizes a performance index subjected to the control force constraint, has been continuously investigated by several authors in optimal control theory [10–12]. The main shortcoming of the bang–bang control becomes clear when one wants to apply this control method to the structural control. First, because control force is not a function of state but of co-state, the on-line computation process of it will significantly increase the time delay and may lead to instability due to the accumulated error in on-line numerical evaluation. Second, the undesirable control chattering near the origin of state space due to high-frequency switching of control force often occurs and great care should also be taken against spillover instability at higher modes. Some studies have been investigated to overcome these shortcomings. Mongkol et al. [5] proposed the linear saturation control method which consists of a low-gain linear control when the system is close to the zero state and the bang–bang control otherwise. They showed a scheme to synthesize the switching surface that is needed to implement the bang–bang control as well. Indrawan et al. [6,13] developed the bound-force control method which excludes the control-effort penalty from the performance index defined in the case of LQR control and defines it at the end of each time interval and seeks the optimal control force for each time interval. Wu and Soong [7] introduced the suboptimal bang–bang control described by a function of state. In the suboptimal bang–bang control, the control force is determined by minimizing the time derivative of a quadratic Lyapunov function under the control force constraint. Wu and Soong [7] also proposed the modified bang–bang control method which overcomes control chattering problem of the suboptimal bang–bang control. This method is found to be effective under a certain range of control force but it can be unstable outside of this range. To overcome this instability, Lim et al. [8] proposed an adaptive bang–bang control algorithm. All the aforementioned bang–bang-type control algorithms guarantee only stability for nominal LTI system and do not explain analytically the robustness with respect to parameter uncertainties of the system.

To explain robust stability of the saturation control, there were some attempts using the property of the robustness of the sliding-mode control (SMC) with respect to parameter uncertainties. Cai et al. [14] proposed the modified sliding-mode bang–bang control method based on a combination of the SMC control and the modified bang–bang control methods. They tried to determine whether there exists a sliding mode on the switching surface. But they reached to the following meaningless result: sliding mode only exists on the switching surface at least near the origin of the system. Yang et al. [9,15] presented the saturated SMC method based on the theory of the SMC and proved it to be an effective method in vibration control for civil building structures. This method can be applicable only to the stable open-loop system. Even though this method is robust with respect to parameter uncertainties of the *system*, it cannot prescribe the bounds of parameter uncertainties of the *system* within which closed-loop robust stability is guaranteed for certain. The complete response of a SMC system consists of two phases of different modes: the reaching mode and the sliding mode. Robustness of the SMC with respect to parameter uncertainties is guaranteed only in the sliding mode. *Therefore the robustness of the SMC is not guaranteed over the complete response of a SMC system.* Furthermore, when the actuators saturate, system trajectories are in the reaching mode for more time than unsaturated system and bounds of parameter uncertainties are narrower than unsaturated system.

To the author's knowledge, studies on robust saturation controller, which analytically address bounds of parameter uncertainties within which closed-loop robust stability of uncertain LTI system is guaranteed in the presence of actuator's saturation, have not been presented. In this paper, we focus on developing robust saturation controller guaranteeing robust stability of uncertain LTI system over the prescribed upper and

lower bounds of structured real parameter uncertainties. In high-rise building, masses, stiffnesses, and damping coefficients are physical system parametric uncertainties. Because these uncertainties are regarded as time-invariant, we can model high-rise building as LTI vibrating system with uncertain masses, stiffnesses, and damping coefficients (or uncertain natural frequencies and damping ratios). Approach of expressing system uncertainties as structured real parameter uncertainties is known to be an effective way of describing the modeling errors in state space. Through this approach, the controller can be designed to guarantee robust stability and/or performance for given *bounds of each parameter uncertainty*. To reduce conservatism of the classical quadratic stability test, both uncertain LTI system and its Lyapunov function are assumed to be affine in time-invariant uncertain real parameters. Suboptimal bang–bang control method, which is designed based on Lyapunov stability condition for nominal LTI system, is extended to this uncertain LTI system. The *robust optimal bang–bang controller* is newly derived by minimizing the time derivative of affine Lyapunov function subjected to the limit of control force. In this case, the affine quadratic stability (AQS) definition and multi-convexity concept [16] to reduce the problem to linear matrix inequality (LMI) problem are introduced. Unfortunately, this controller is gain-scheduled type and its implementation requires the exact knowledge of uncertain parameters. A robust saturation controller with fixed gain, which does not require the knowledge of uncertain parameters, is proposed by modifying this controller. *LMI-based* sufficient existence conditions are presented to design this proposed controller. A practical numerical example is illustrated to verify the availability and the effectiveness of the proposed controller. It is shown through numerical simulations that the proposed robust saturation controller with a fixed gain is robustly stable with respect to parameter uncertainties over the prescribed upper and lower bounds.

2. Affine quadratic stability

We first review the AQS test which is used as an analytical tool for our robust saturation control. Even though it is much more difficult to deal with its analysis mathematically, the AQS test can analyze robust stability of linear systems with uncertain real parameters which are time-invariant or time-varying. In robust controller design, we can obtain robust stability tests for systems under various model uncertainty assumptions through the use of various stability analyses. Bounds of parameter uncertainties are different according to how to represent model uncertainties and how to approach stability analysis. The classical quadratic stability test guarantees robust stability against arbitrarily fast parameter variations [17]. As a result, this quadratic stability test can be very conservative for constant parameters or slow-varying parameters. However, the AQS test is applicable to both constant and time-varying uncertain parameters and much less conservative than the quadratic stability test in the case of constant parameters or slow-varying parameters. This AQS test is an extension of the notion of quadratic stability where the fixed quadratic Lyapunov function is replaced by a Lyapunov function with affine dependence on uncertain parameters. This paper is concerned with the LTI system with constant uncertain real parameters that can be described by state space equation of the form

$$\dot{x}(t) = A(\theta)x(t), \quad x(0) = x_0, \quad (1)$$

where state vector is $x \in \mathbf{R}^n$, $\theta = (\theta_1, \theta_2, \dots, \theta_K) \in \mathbf{R}^K$ is a vector of uncertain real parameters, and the system matrix $A(\theta)$ is assumed to be stable and depends affinely on the parameters of θ_i . That is

$$A(\theta) = A_0 + \theta_1 A_1 + \theta_2 A_2 + \dots + \theta_K A_K, \quad (2)$$

where $A_0, A_1, A_2, \dots, A_K$ are known fixed matrices.

We assume that lower and upper bounds are available for the parameter values. Specifically, each parameter θ_i ranges between known external values $\underline{\theta}_i$ and $\overline{\theta}_i$.

$$\theta_i \in [\underline{\theta}_i, \overline{\theta}_i] \quad \text{for } i = 1, 2, \dots, K. \quad (3)$$

This means that the parameter vector θ is valued in a hyper-rectangle called the parameter box. In the sequel

$$\Theta := \{(\omega_1, \omega_2, \dots, \omega_K) : \omega_i \in [\underline{\theta}_i, \overline{\theta}_i]\} \quad (4)$$

denotes the set of the 2^K vertices or corners of these parameters.

The following notion of parameter-dependent Lyapunov function is introduced to reduce conservatism of the classical quadratic stability test when system (1) is affine in θ with time-invariant parameters.

$$V(x, \theta) = x^T P(\theta)x, \quad (5)$$

where $P(\theta)$ is an affine function of θ .

$$P(\theta) = P_0 + \theta_1 P_1 + \theta_2 P_2 + \cdots + \theta_K P_K. \quad (6)$$

Using this parameter-dependent Lyapunov function we can define AQS for the LTI systems with constant uncertain real parameters (1) as follows.

Definition 1. (AQS, Gahinet et al. [16]). The LTI system with constant uncertain real parameters (1) is said to be affinely quadratically stable if there exist $K + 1$ symmetric matrices $P_0, P_1, P_2, \dots, P_K$ such that

$$P_0 + \theta_1 P_1 + \theta_2 P_2 + \cdots + \theta_K P_K > 0, \quad (7)$$

$$A(\theta)^T P(\theta) + P(\theta)A(\theta) < 0 \quad (8)$$

hold for all admissible trajectories of the parameter vector $\theta = (\theta_1, \theta_2, \dots, \theta_K)$.

Definition 1 expresses that $V(x, \theta) > 0$ and $dV(x, \theta)/dt < 0$ for all admissible parameter trajectories. Recall that the AQS test is much less conservative than the classical quadratic stability test seeking a fixed parameter-independent Lyapunov function along all admissible parameter trajectories. By imposing additional ‘‘multi-convexity’’ constraints on the parameter-dependent Lyapunov function, finding an affine Lyapunov matrix $P(\theta)$ can be turned into an LMI problem with variables $P_0, P_1, P_2, \dots, P_K$ [16]. Efficient polynomial-time optimization algorithms are available to solve this [18] because LMI problems are convex. Note that more general definition of AQS can be handled for uncertain linear system with time-varying parameter uncertainties by adding lower and upper bounds on rate of variation of parameter uncertainties [16].

3. Gain-scheduled robust optimal bang–bang controller

We focus on designing robust saturation controller, which guarantees stability of uncertain LTI system with actuator saturation, over the prescribed upper and lower bounds of structured real parameter uncertainties analytically. Suboptimal bang–bang control method [7], which is designed based on Lyapunov stability condition for nominal LTI system, is extended to uncertain LTI system. To reduce conservatism of classical quadratic stability, both uncertain LTI system (1) and its Lyapunov function (5) are affine in time-invariant uncertain real parameters. In this section, the *robust optimal bang–bang controller* is derived by applying the AQS definition and minimizing the time derivative of affine Lyapunov function subjected to the limit of control force. To focus our attention on designing robust saturation controller based on AQS, we add control force term to Eq. (1). We consider the following uncertain LTI system:

$$\dot{x}(t) = A(\theta)x(t) + Bu(t), \quad x(0) = x_0 \quad (9)$$

with control force constraint.

$$|u(t)| \leq u_{\max}, \quad (10)$$

where B is control input vector, and u is scalar control force.

For uncertain LTI system (9), a parameter-dependent Lyapunov function (5) is defined. The time derivative of this Lyapunov function is of the following form:

$$\dot{V}(x, \theta) = x^T [A(\theta)^T P(\theta) + P(\theta)A(\theta)]x + 2x^T P(\theta)Bu. \quad (11)$$

The optimal control force of minimizing this time derivative of parameter-dependent Lyapunov function under the control force constraint (10) takes the form

$$u(t) = -u_{\max} \cdot \text{sgn}[B^T P(\theta)x(t)], \quad (12)$$

where $P(\theta)$ satisfies the following equations which correspond to $2^{K+1} + K$ LMI conditions:

$$P(\omega) > 0 \quad \text{for all } \omega \in \Theta, \tag{13}$$

$$A(\omega)^T P(\omega) + P(\omega) A(\omega) < 0 \quad \text{for all } \omega \in \Theta, \tag{14}$$

$$A_i^T P_i + P_i A_i \geq 0 \quad \text{for } i = 1, 2, \dots, K. \tag{15}$$

Since both $A(\theta)$ and $P(\theta)$ are affine in θ , assessing whether system (1) is affinely quadratically stable is not tractable in general, neither analytically nor numerically. In particular, it is no longer sufficient to check Eqs. (13) and (14) at the vertices of the parameter box because Eq. (14) is a nonconvex problem. To reduce Eq. (14) to a finite set of LMI constraints, we must further restrict the choice of affine Lyapunov matrix $P(\theta)$. The additional ‘‘multi-convexity’’ constraint (15) reduces the problem of finding affine parameter-dependent Lyapunov matrices to an LMI problem [16].

This *robust optimal bang–bang controller* is gain-scheduled type. Note that the robust *optimal bang–bang controller* derived by applying classical quadratic stability is not gain-scheduled type but a fixed-gain type because this seeks a fixed parameter-independent Lyapunov function. As in the case of linear controller design method using parameter-dependent Lyapunov function [19–21], we know that this gain-scheduled *robust optimal bang–bang controller* derived by applying AQS makes bounds of parameter uncertainties much broader than those of a fixed-gain controller derived by applying quadratic stability.

4. Robust saturation controller with a fixed gain

Because the optimal control force in Eq. (12) is the function of uncertain parameters θ , the *robust optimal bang–bang controller* is a gain-scheduled type. In controller (12), its implementation requires the exact knowledge of uncertain parameters. But we only know upper and lower bounds of uncertain parameters instead of exact values. So it is difficult and impractical to apply this gain-scheduled *robust optimal bang–bang controller* to a real practical system. In this section, we propose the *LMI-based* sufficient conditions for the existence of robust saturation controller with a fixed gain instead of gain-scheduled.

To formulate new controller, Eq. (12) can be expressed by using saturation function instead of sign function.

$$u(t) = -\text{sat}[\delta B^T P_0 x(t)], \tag{16}$$

where $\delta > 0$ and P_0 satisfies Eqs. (13)–(15) which correspond to $2^{K+1} + K$ LMI conditions.

This robust saturation controller (16) has similar form to the suboptimal bang–bang controller ($u(t) = -u_{\max} \cdot \text{sgn}[B^T P_0 x(t)]$) for nominal LTI system and is expressed by using saturation function instead of sign function. Note that using saturation function is a typical choice to overcome control chattering problem occurring in sign-function-type controllers.

We can express Eq. (16) in the following form introducing $\beta(x(t))$:

$$\begin{aligned} u(t) &= -\beta(x(t)) \cdot \delta B^T P_0 x(t), \\ \beta(x(t)) &= \frac{\text{sat}(\delta B^T P_0 x(t))}{\delta B^T P_0 x(t)}, \\ \beta(x(t)) &= 1 \quad \text{if } B^T P_0 x(t) = 0, \end{aligned} \tag{17}$$

where $0 < \beta(x(t)) \leq 1$.

Along the trajectories of system (9) with the controller given in Eq. (17), the time derivative of $V(x, \theta)$ in Eq. (5) is obtained

$$\begin{aligned} \dot{V}(x, \theta) &= x^T [A(\theta)^T P(\theta) + P(\theta) A(\theta)] x \\ &+ x^T \left[-\beta \cdot \delta \cdot \left\{ 2P_0 B B^T P_0 + \sum_{i=1}^K \theta_i (P_0 B B^T P_i + P_i B B^T P_0) \right\} \right] x. \end{aligned} \tag{18}$$

To show $\dot{V}(x, \theta) < 0$, we first seek $K + 1$ symmetric matrices $P_0, P_1, P_2, \dots, P_K$ which satisfy that the first term in the right-hand term of Eq. (18) is less than 0, and then substitute the found matrices into the second term in the right-hand term of Eq. (18). Robust stability of the saturation controller (16) is guaranteed if the second term in the right-hand term of Eq. (18) is less than 0 with $\delta > 0$. But it is not guaranteed that the second term in the right-hand term of Eq. (18) is less than 0 with $\delta > 0$. Unfortunately, we cannot guarantee robust stability of controller (16) in this way. To guarantee robust stability of controller (16) we propose the following theorem which gives sufficient conditions for the existence of robust saturation controller with a fixed gain.

Theorem 1. Consider an uncertain LTI system (9) where $A(\theta)$ depends affinely on the parameter vector $\theta = (\theta_1, \theta_2, \dots, \theta_K)$, θ_i satisfies Eq. (3), and control force has constraint of Eq. (10). Let Θ denote the sets of vertices of the parameter box Eq. (4). Robust stability of the saturation controller (16) is guaranteed if there exist $K + 1$ symmetric matrices $P_0, P_1, P_2, \dots, P_K$, and positive-definite symmetric matrix M_a satisfying Eqs. (13), (15), and (19), and if there exists $\delta > 0$ which satisfies Eq. (20) for these matrices $P_0, P_1, P_2, \dots, P_K$ and M_a .

$$A(\omega)^T P(\omega) + P(\omega)A(\omega) + M_a < 0 \quad \text{for all } \omega \in \Theta, \quad (19)$$

$$M_a + \delta \cdot \left\{ 2P_0 B B^T P_0 + \sum_{i=1}^K \theta_i (P_0 B B^T P_i + P_i B B^T P_0) \right\} > 0 \quad \text{for all } \omega \in \Theta. \quad (20)$$

Proof. The following equation is obtained from adding and subtracting M_a each term in the right-hand term of Eq. (18):

$$\begin{aligned} \dot{V}(x, \theta) = & x^T [A(\theta)^T P(\theta) + P(\theta)A(\theta) + M_a]x \\ & + x^T \left[-M_a - \beta \cdot \delta \cdot \left\{ 2P_0 B B^T P_0 + \sum_{i=1}^K \theta_i (P_0 B B^T P_i + P_i B B^T P_0) \right\} \right] x. \end{aligned} \quad (21)$$

We assume that there exist symmetric matrices $P_0, P_1, P_2, \dots, P_K$, and M_a satisfying that the first term in the right-hand term of Eq. (21) is less than 0. Robust stability of the saturation controller of Eq. (16) is guaranteed if the second term in the right-hand term of Eq. (21) is less than 0 with $\delta > 0$ when we substitute these matrices into the second term in the right-hand term of Eq. (21). The first term in the right-hand term of Eq. (21) is always less than 0 if there exist symmetric matrices $P_0, P_1, P_2, \dots, P_K$, and positive-definite symmetric matrix M_a satisfying Eqs. (13), (15), and (19). Let $\rho_i = \delta \theta_i$ ($i = 1, 2, \dots, K$), then $\delta \underline{\theta}_i \leq \rho_i \leq \delta \bar{\theta}_i$ and the second term in the right-hand term of Eq. (21) is rewritten as follows:

$$-x^T \left[M_a + \beta \left\{ \delta (2P_0 B B^T P_0) + \sum_{i=1}^K \rho_i (P_0 B B^T P_i + P_i B B^T P_0) \right\} \right] x. \quad (22)$$

Here we first consider the case of $\beta = 1$. For given $\delta > 0$, the following LMI of Eq. (23) is a convex constraint on the variables ρ_i because M_a , $P_0 B B^T P_0$, and $P_0 B B^T P_i + P_i B B^T P_0$ are the symmetric matrices respectively:

$$M_a + \delta (2P_0 B B^T P_0) + \sum_{i=1}^K \rho_i (P_0 B B^T P_i + P_i B B^T P_0) > 0. \quad (23)$$

When we define Φ as the set of the 2^K vertices of ρ_i , Eq. (23) is satisfied for all ρ_i if and only if Eq. (23) is satisfied in Φ by convexity of Eq. (23).

$$\Phi := \{(\psi_1, \psi_2, \dots, \psi_K) : \psi_i \in \{\delta \underline{\theta}_i, \delta \bar{\theta}_i\}\}. \quad (24)$$

Eq. (23) is equivalent to Eq. (20). Next we consider the case of $0 < \beta < 1$. For given $\delta > 0$, we can easily show that Eq. (22) is less than 0 if Eq. (23) is satisfied. Therefore, Eq. (22) is always less than 0 if Eq. (20) is satisfied for given $\delta > 0$. \square

Remark 1. In Eq. (19), we must solve LMIs with $K + 2$ variables including matrix M_a . To reduce numerical computational burden we can reduce them to LMIs with $K + 1$ variables by setting M_a in an arbitrary matrix. And we can seek δ_{\max} by setting δ in a fixed value and sweeping through δ in Eq. (20), because the maximum value of δ satisfying Eq. (20) is finite ($0 < \delta \leq \delta_{\max}$). δ is used to maximize the utilization of the available control force. The control performance can be improved by using large δ .

Remark 2. There is a difficulty in the numerical implementation of Theorem 1. It comes from the multi-convexity constraint of Eq. (15). As suggested in Ref. [16], we can relax the multi-convexity of the function $dV(x, \theta)/dt < 0$ by only requiring that it is bounded by a multi-convex function. In this case, for nonnegative definite symmetric matrix N_i ($i = 1, 2, \dots, K$) LMI conditions of Eqs. (15) and (19) are replaced by the following equations:

$$A(\omega)^T P(\omega) + P(\omega)A(\omega) + M_a + \sum_{i=1}^K \omega_i^2 N_i < 0 \quad \text{for all } \omega \in \Theta, \quad (25)$$

$$A_i^T P_i + P_i A_i + N_i \geq 0 \quad \text{for } i = 1, 2, \dots, K, \quad (26)$$

$$N_i \geq 0 \quad \text{for } i = 1, 2, \dots, K. \quad (27)$$

A simple remedy is to choose $N_i = \lambda_i I$ with $\lambda_i > 0$.

While it is not optimal any longer, this proposed robust saturation controller does not require the exact values of the uncertain parameters. So it is easy and practical to apply this robust saturation controller to a real practical system.

5. Numerical example

In this section, a practical numerical example for a linear building is illustrated to verify the feasibility of the proposed robust saturation controller with a fixed gain (16) and simulation results are presented. LMIs in Theorem 1 are solved using Matlab[®] and LMI control toolbox [22]. Controller design parameter M_a is chosen at an arbitrary value by trial and error as suggested in Remark 1.

A three-story scaled building model studied by Kobori and Kamagata [23], Yang et al. [9], and Cai et al. [14], in which every story unit is identically constructed and an active brace system (ABS) is installed in the first-story unit, as shown in Fig. 1, is considered. The mass, stiffness and damping coefficient of each story unit for nominal system are $m_i = 1000$ kg, $k_i = 980$ kN/m, and $c_i = 1.407$ kN s/m, respectively, for $i = 1, 2, 3$. The El Centro earthquake (north–south component, 1940) scaled to a maximum acceleration of $0.112g$ is used as the input excitation. The maximum control force u_{\max} is 700 N and uncertainties of the system are stiffnesses and damping coefficients of each floor. Let uncertainties of stiffnesses be θ_i ($i = 1, 2, 3$) and the uncertainties of damping coefficients θ_i ($i = 4, 5, 6$), then the admissible trajectories are given by $k_i(1 + \theta_i)$ for $i = 1, 2, 3$ and $c_{i-3}(1 + \theta_i)$ for $i = 4, 5, 6$ specified in multiplicative form. This uncertain system can be described by state space equation as follows:

$$\dot{x}(t) = A(\theta)x(t) + Bu(t) + E\ddot{x}_g, \quad (28)$$

where x_i ($i = 1, 2, 3$) are the relative displacement of each floor to ground, state vector $x = [x_1 \ x_2 \ x_3 \ \dot{x}_1 \ \dot{x}_2 \ \dot{x}_3]^T$, control input vector $B = [0 \ 0 \ 0 \ 1/m_1 \ 0 \ 0]^T$, the disturbance input vector $E = [0 \ 0 \ 0 \ -1 \ -1 \ -1]^T$, and uncertain system matrix $A(\theta)$ is

$$A(\theta) = A_0 + \theta_1 A_1 + \theta_2 A_2 + \theta_3 A_3 + \theta_4 A_4 + \theta_5 A_5 + \theta_6 A_6, \quad (29)$$

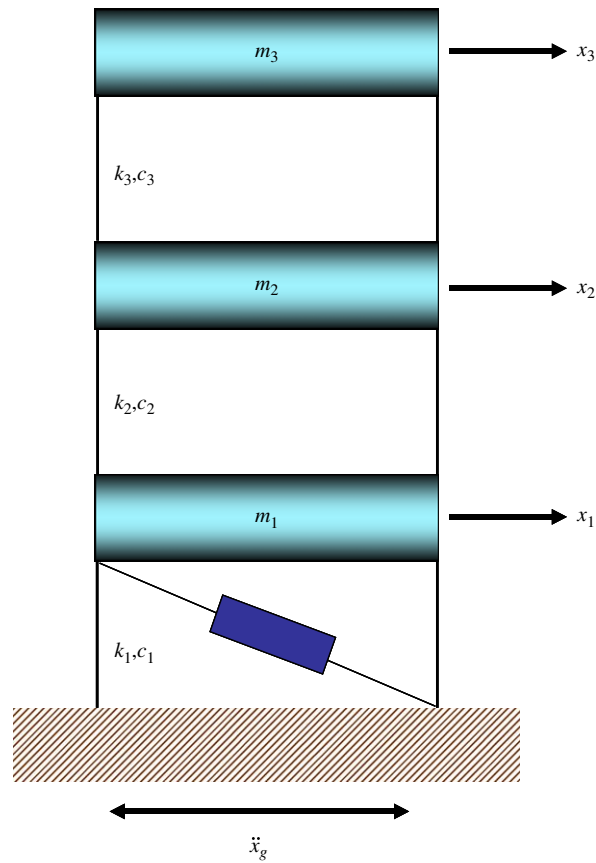


Fig. 1. Three-story building model with ABS.

where

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{k_1 + k_2}{m_1} & \frac{k_2}{m_1} & 0 & -\frac{c_1 + c_2}{m_1} & \frac{c_2}{m_1} & 0 \\ \frac{k_2}{m_2} & -\frac{k_2 + k_3}{m_2} & \frac{k_3}{m_2} & \frac{c_2}{m_2} & -\frac{c_2 + c_3}{m_2} & \frac{c_3}{m_2} \\ 0 & \frac{k_3}{m_3} & -\frac{k_3}{m_3} & 0 & \frac{c_3}{m_3} & -\frac{c_3}{m_3} \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{k_1}{m_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{c_1}{m_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{k_2}{m_1} & \frac{k_2}{m_1} & 0 & 0 & 0 & 0 \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{c_2}{m_1} & \frac{c_2}{m_1} & 0 \\ 0 & 0 & 0 & \frac{c_2}{m_2} & -\frac{c_2}{m_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{k_3}{m_2} & \frac{k_3}{m_2} & 0 & 0 & 0 \\ 0 & \frac{k_3}{m_3} & -\frac{k_3}{m_3} & 0 & 0 & 0 \end{bmatrix}, \quad A_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{c_3}{m_2} & \frac{c_3}{m_2} \\ 0 & 0 & 0 & 0 & \frac{c_3}{m_3} & -\frac{c_3}{m_3} \end{bmatrix}.$$

Before we confirm the robust stability of the proposed controller, we ascertain general trends of the proposed controller according to the value of the controller design parameter M_a . For convenience we replace M_a by $\mu_a I$ with $\mu_a > 0$. Firstly, we investigated bounds of parameter uncertainties according to μ_a . Let $\underline{\theta} \leq \theta_i \leq \bar{\theta}$ ($i = 1, 2, \dots, 6$) for all parameter uncertainties. We can examine some characteristics of the proposed controller through Table 1. Table 1 shows allowable stable bounds of uncertainties θ_i of the proposed controller with a fixed gain according to μ_a and compares them with those of the gain-scheduled controller. The case of $\mu_a = 0$ corresponds to the gain-scheduled controller. Introducing μ_a to design the fixed-gain controller makes bounds of parameter uncertainties narrower than those of the gain-scheduled controller. And the larger μ_a is, the narrower bounds of uncertainties θ_i are. Secondly, we investigated control performance according to μ_a . The control performance can be improved by using large δ because δ is used to maximize the utilization of the available control force. Let $|\theta_i| \leq \theta_e$ ($i = 1, 2, \dots, 6$) for all parameter uncertainties. Here we set θ_e arbitrarily. Table 2 shows values of δ_{\max} according to μ_a in the case of $\theta_e = 0.4$. The larger μ_a is, the larger δ_{\max} is. So the larger μ_a makes the control performance better because δ_{\max} is closely related to the control performance of the proposed controller. Note that $\delta_{\max} = 0$ in the case of $\mu_a = 0$. This means that the second term in the right-hand term of Eq. (18) is not less than 0 with $\delta > 0$. However, introducing the controller design parameter M_a can make $\delta > 0$ as can be seen in Eq. (20).

Next, we confirm the robust stability and the effectiveness of the proposed controller. For given bounds of uncertain parameters $\theta_e = 0.4$, the controller design parameter $M_a = \text{diag}(5e5, 5e5, 5e5, 5e2, 5e2, 5e2)$ is chosen for good performance of the controller. The computed value of δ_{\max} is about 519. We designed the robust saturation controller (16) with $\delta = 519$. Control performance of the proposed controller is compared with other controllers from the viewpoint of maximum responses (maximum interstory drifts d_i and maximum absolute accelerations \ddot{x}_{ai}).

Table 1
Bounds of uncertainties θ_i according to μ_a

μ_a	$\underline{\theta}$	$\bar{\theta}$
0	-0.99	175
10	-0.99	169
50	-0.99	161
100	-0.99	158
500	-0.99	125

Table 2
Values of δ_{\max} according to μ_a ($|\theta_i| \leq 0.4$)

μ_a	δ_{\max}
0	0
5e1	0.9
5e2	1.1
1e3	2.1
5e3	4.2
1e4	21.6
5e4	88.3
6e4	104.5

Table 3
Maximum response values for nominal system

Story	No control		LQR		MBBC		SSMC		Proposed	
	d_i (cm)	\ddot{x}_{ai} (m/s ²)	d_i (cm)	\ddot{x}_{ai} (m/s ²)	d_i (cm)	\ddot{x}_{ai} (m/s ²)	d_i (cm)	\ddot{x}_{ai} (m/s ²)	d_i (cm)	\ddot{x}_{ai} (m/s ²)
1	1.34	3.13	0.88	2.25	0.51	1.64	0.52	1.62	0.53	1.60
2	1.02	4.75	0.66	3.24	0.47	2.27	0.49	2.27	0.48	2.20
3	0.60	5.84	0.35	3.41	0.32	3.20	0.33	3.27	0.32	3.17

For nominal system, Table 3 shows control performance of the proposed controller on maximum response values of the system in comparison with those of the classical LQR controller, the modified bang–bang controller (MBBC), and the saturated sliding mode controller (SSMC). The LQR controller is adjusted so that maximum control force is about 700 N. It is observed from Table 3 that control performances of saturation controllers such as the MBBC, the SSMC, and the proposed controller are quite remarkable in comparison with that of the LQR controller. Time histories for absolute acceleration of the third-story unit, drift of the first-story unit, and control force using the LQR controller and the proposed controller are presented in Fig. 2 in comparison with the responses without control. Saturation controllers considering the limit of control force are known to be more effective than the LQR controller in maximum response reduction under the same maximum control force (see Ref. [7] for the MBBC and Ref. [9] for the SSMC). The proposed controller also produces better performance than the LQR controller in terms of maximum response reduction under the same maximum control force as shown in Table 3 and Fig. 2. In the reduction of interstory drifts, the MBBC is the most effective and the proposed controller and the SSMC have almost the same effectiveness.

For uncertain system, we ascertained that our proposed controller guarantees robust stability within all the range of parameter uncertainties considered in controller design. Robust stability of the proposed controller, which is guaranteed in Theorem 1 analytically, was verified through numerical simulations for the cases with various parameter uncertainties. Among them, Fig. 3 shows responses (absolute acceleration of the third-story unit and drift of the first-story unit) of the system and control force in the case of parameter uncertainties with $\theta_1 = \theta_2 = \theta_3 = 0.4$ and $\theta_4 = \theta_5 = \theta_6 = -0.4$, and Table 4 shows control performance of the proposed controller on maximum response values of the system in comparison with that of the SSMC in the case of parameter uncertainties with $\theta_i = 0.4$ ($i = 1, 2, \dots, 6$) and $\theta_i = -0.4$ ($i = 1, 2, \dots, 6$). The control performances of these two controllers are almost similar. Through extensive numerical simulations, it is checked within considered bounds of parameter uncertainties that the proposed controller has almost the same effectiveness in maximum responses reduction in comparison with the SSMC. But from the viewpoint of robust stability, only the proposed controller can address bounds of parameter uncertainties analytically within which robust stability is guaranteed over the complete response of system.

Our proposed controller uses saturation function instead of sign function. And for given bounds of parameter uncertainties, maximum value of δ guaranteeing robust stability of this controller is finite. The larger bounds of parameter uncertainties are, the smaller maximum value of δ is. So the slope of saturation

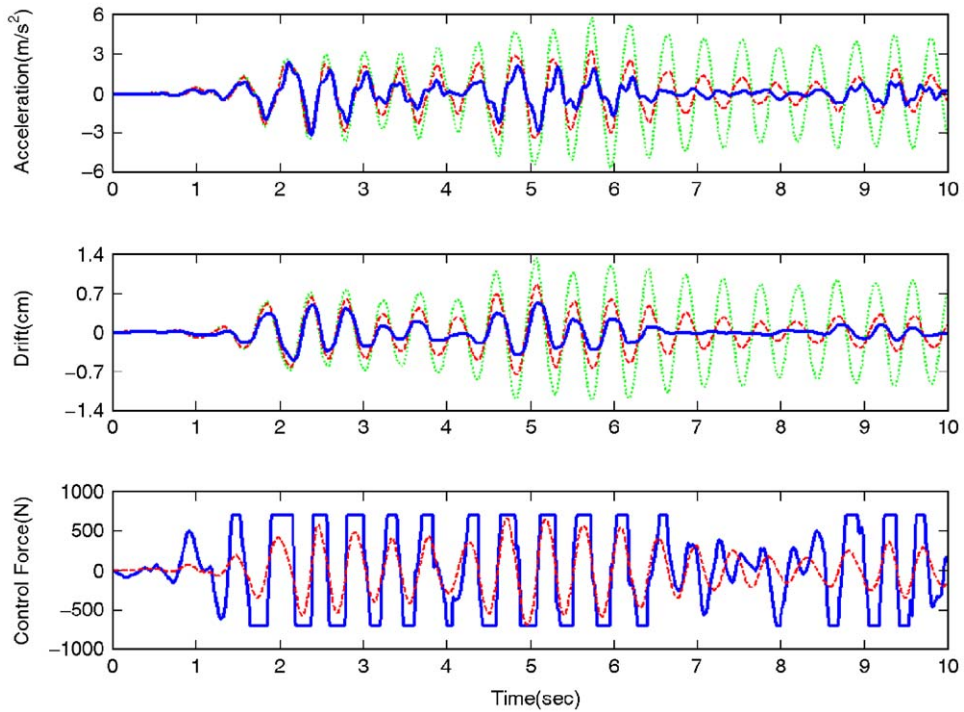


Fig. 2. Comparison of responses and control forces for nominal system applying the LQR controller and the proposed controller (····, no control; ----, LQR controller; —, proposed controller).

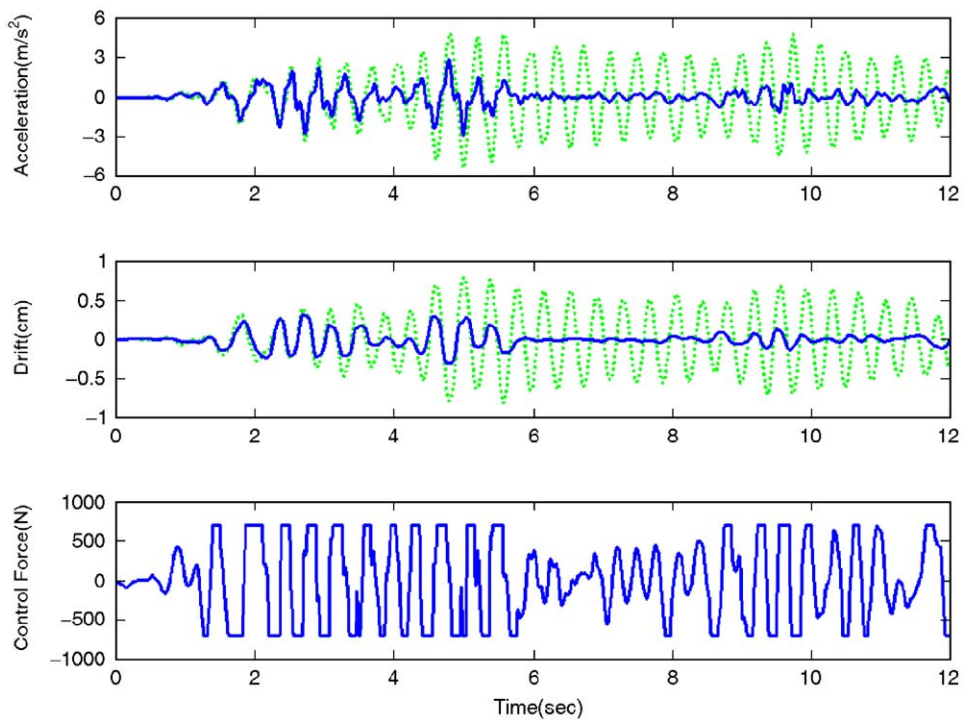


Fig. 3. Time histories of responses for uncertain system with parameter uncertainties ($\theta_1 = \theta_2 = \theta_3 = 0.4$ and $\theta_3 = \theta_4 = \theta_5 = -0.4$) applying the proposed controller (····, no control; ----, proposed controller).

Table 4
Maximum response values for uncertain system

Story	$\theta_i = 0.4 (i = 1, 2, \dots, 6)$						$\theta_i = -0.4 (i = 1, 2, \dots, 6)$					
	No control		SSMC		Proposed		No control		SSMC		Proposed	
	d_i (cm)	\ddot{x}_{ai} (m/s ²)	d_i (cm)	\ddot{x}_{ai} (m/s ²)	d_i (cm)	\ddot{x}_{ai} (m/s ²)	d_i (cm)	\ddot{x}_{ai} (m/s ²)	d_i (cm)	\ddot{x}_{ai} (m/s ²)	d_i (cm)	\ddot{x}_{ai} (m/s ²)
1	0.70	2.11	0.31	1.86	0.31	1.82	1.64	2.82	1.05	1.91	1.06	2.01
2	0.57	3.39	0.28	1.91	0.28	1.94	1.34	3.45	0.97	2.37	0.97	2.47
3	0.34	4.64	0.20	2.69	0.19	2.64	0.79	4.63	0.61	3.60	0.61	3.59

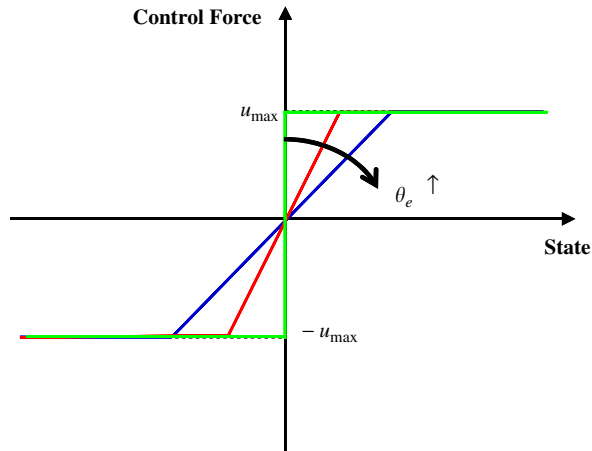


Fig. 4. The slopes of saturation function of the proposed controller according to given bounds of parameter uncertainties.

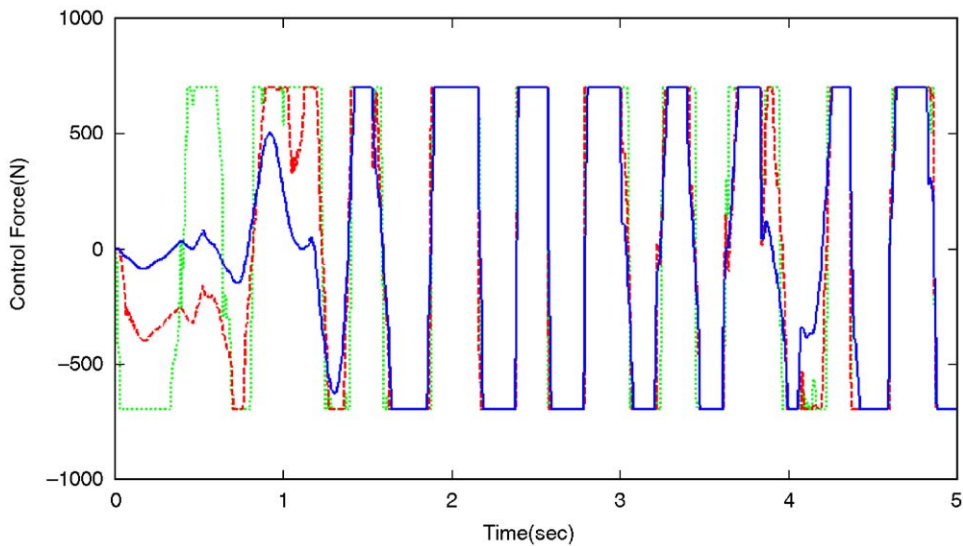


Fig. 5. Comparison of control forces of the proposed controller for nominal system according to given bounds of parameter uncertainties (···, $\theta_e = 0.0$; ----, $\theta_e = 0.2$; —, $\theta_e = 0.4$).

function is gentler as bounds of parameter uncertainties are larger as shown in Fig. 4. This trend can be checked through Fig. 5 in our example. Fig. 5 shows control forces of three cases ($\theta_e = 0, 0.2,$ and 0.4) for nominal system. The case of $\theta_e = 0$ shows bang–bang-type control force of the MBBC because it corresponds to nominal system. The slope of $\theta_e = 0.4$ is gentler than that of $\theta_e = 0.2$ because of smaller maximum values of δ . Therefore, our proposed controller moves away further from bang–bang-type controller and its control performance may be worse in comparison with the MBBC for nominal system as bounds of parameter uncertainties are larger. However, we can guarantee robust stability at the cost of performance degradation.

In actual civil engineering structural control, one of the inevitable problems in large-scale practical applications is time delay. Servo-hydraulic actuators and servo-motors are often used as control force devices [3]. Time delay exists obviously in these actuators and it results in unsynchronized control force applied to the structures. Neglecting this time delay may even render the control system unstable. There are some methods to overcome time delay problem. One is to design controller using mathematical model in which actuator's dynamics are included [8,24]. And another method is to include time delay in controller design [25–27].

6. Conclusions

The objective of this paper is to develop robust saturation controller guaranteeing *robust* stability of uncertain LTI system over the prescribed upper and lower bounds of structured real parameter uncertainties analytically. Based on AQS and multi-convexity concept, the *robust optimal bang–bang controller* was newly derived by minimizing the time derivative of affine Lyapunov function subjected to the limit of control force. This controller guarantees the closed-loop robust stability of the system within bounds of parameter uncertainties in the presence of actuator's saturation. The bounds of parameter uncertainties in this controller are much broader than those in the *robust optimal bang–bang controller* based on quadratic stability. Since this controller is gain-scheduled controller type, it requires the exact knowledge of uncertain parameters. Therefore, it is impractical to apply this controller to real systems. To overcome this shortcoming, another robust saturation controller with a fixed gain was proposed by modifying *this robust optimal bang–bang controller*. While the new controller is not optimal any longer, it does not require the knowledge of uncertain parameters. Theorem 1 suggested in this paper gives the *LMI-based* sufficient conditions for the existence of this fixed-gain controller by introducing controller design parameter M_a .

Some characteristics of the proposed controller were examined through numerical simulations. Introducing the controller design parameter M_a to design the fixed-gain controller makes bounds of parameter uncertainties narrower than those of the gain-scheduled *robust optimal bang–bang controller*. Also, the larger the bounds of parameter uncertainties are, the smaller the maximum value of δ in controller (16) is. Therefore, while the proposed controller guarantees robust stability within bounds of parameter uncertainties, its control performance may be worse in comparison with any other saturation controllers for nominal system. Note that to guarantee the robust stability for the system uncertainties, which is inevitable in real world, sacrificing the performance a little can be sensible and practical.

The availability and the effectiveness of the proposed controller were also verified through numerical simulations. Simulation results show that the proposed controller is robustly stable with respect to parameter uncertainties over the prescribed upper and lower bounds and the proposed controller can be easily applicable for civil engineering structures.

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